Lecture-1: Different notions of ∞-categories

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Outline

1. Overview of literature and key ideas
   - Lurie's *Higher Topos theory*
   - Pridham, *An introduction to derived (algebraic) geometry,* (2023)
   - Ravenel, *What is an ∞-category?* (2023)
2. ∞-categories as topological categories
3. ∞-categories as simplicial categories

An ∞-category $\mathcal{C}$ is a generalization of an ordinary category, (aka a 1-category).

- It has objects and morphisms (in the sense of usual categories), but additionally there are k-morphisms for $k > 1$, leading to "morphisms between morphisms"-type structure.

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- It has objects and morphisms (in the sense of usual categories), but additionally there are k-morphisms for $k > 1$, leading to "morphisms between morphisms"-type structure.
- Composition is not well-defined (always up to something!)
- Suitable setting to do homotopy theory
- Mapping spaces $Hom_{\mathcal{C}}(X, Y)$ are more structured than sets!

$\exists$ many equivalent descriptions of ∞-categories:

- Useful to have different ways (depending on your purpose)
- We focus on the more accessible ones:

A *Bluffer's Guide*

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(3) Easiest to build: $\underbrace{\textbf{Relative categories}}_{\text{pairs } (\mathcal{C},\mathcal{W}) \text{ where } \mathcal{C} \text{ is a category, } \mathcal{W} \text{ is a subcategory}}$

Relative category description

- $\{f \in Hom_{\mathcal{W}}\} \leftrightarrow$ {equivalence weaker than isomorphism}
- Examples: $\mathcal{W} = \begin{cases} \text{homotopy equivalences for Top} \\ \text{quasi-isomorphisms for Ch} \\ \text{quasi-isomorphisms for } \mathrm{cdga}_{\mathbb{K}}^{\leq 0} \end{cases}$

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- Very little data needed to specify an ∞-category
- Once a notion of weak equivalence is chosen, the rest is determined.
- Can use *model categories* as well.

Definition

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- $\mathcal{C}$: any category with limits/colimits. All morphisms are both fibrations and cofibrations; the weak equivalences are just the isomorphisms.
- For $\mathcal{C} = \mathrm{cdga}_{\mathbb{K}}^{\leq 0}$
$$\begin{cases} \text{weak equivalences} = \text{quasi-isomorphisms} \\ \text{fibrations} = \text{degreewise surjections} \\ \text{cofibrations} = \text{the maps having LLP wrt. trivial fib.} \end{cases}$$
- For $\mathcal{C} = \mathrm{Top}$
$$\begin{cases} \text{weak equivalences} = \text{homotopy equivalences} \\ \text{fibrations} = \text{the maps having HLP} \\ \text{cofibrations} = \text{the maps having HEP} \end{cases}$$

Next: Two (equivalent!) approaches via topological spaces or simplicial sets...